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Applying Hamilton's vector formula in mathematics education: enhancing student mathematical skills through innovative teaching methods

Aplicação da fórmula vetorial de Hamilton na educação matemática: aprimorando as habilidades matemáticas dos alunos por meio de métodos de ensino inovadores

Aplicando la fórmula vectorial de Hamilton en la educación matemática: mejorando las habilidades matemáticas en estudiantes mediante métodos de enseñanza innovadores

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ABSTRACT

The relevance of this discussion is driven by the need to improve mathematics teaching methods to enhance students' mathematical competencies, particularly critical thinking and problem-solving skills. With its broad didactic potential, Hamilton's vector formula becomes an essential tool in the educational process. The study aims to examine the impact of integrating Hamilton's vector formula on the development of students' mathematical competencies. The research methodology includes a literature review, expert evaluations, a pedagogical experiment with control and experimental groups, testing, and student surveys. The study results showed that using Hamilton's vector formula promotes the development of students' critical and analytical thinking, increases their confidence in solving complex problems, and improves overall mathematical competencies. The experimental group demonstrated higher test scores and practical task performance than the control group. Students noted that working with Hamilton's vector formula helped them better understand geometric concepts and see the practical application of theoretical knowledge. The survey showed a positive perception of the new methodology, although some initial difficulties were noted. The practical significance of the results lies in developing recommendations for the effective use of Hamilton's vector formula in the educational process, which can significantly improve the quality of students' mathematical training and contribute to their successful learning process in the future.

Keywords: Hamilton's formula, Euler formula, orthocenter, homothety, mathematics teaching, applied mathematics.

RESUMO

A relevância deste tema é impulsionada pela necessidade de melhorar os métodos de ensino da matemática para aprimorar as competências matemáticas dos alunos, particularmente as habilidades de pensamento crítico e resolução de problemas. Com seu amplo potencial didático, a fórmula vetorial de Hamilton torna-se uma ferramenta essencial no processo educacional. O estudo visa examinar o impacto da integração da fórmula vetorial de Hamilton no desenvolvimento das competências matemáticas dos alunos. A metodologia de pesquisa inclui revisão de literatura, avaliações de especialistas, um experimento pedagógico com grupos de controle e experimental, testes e questionários aos alunos. Os resultados do estudo mostraram que o uso da fórmula vetorial de Hamilton promove o desenvolvimento do pensamento crítico e analítico dos alunos, aumenta sua confiança na resolução de problemas complexos e melhora as competências matemáticas gerais. O grupo experimental demonstrou pontuações mais altas em testes e desempenho em tarefas práticas do que o grupo de controle. Os alunos notaram que trabalhar com a fórmula vetorial de Hamilton os ajudou a entender melhor os conceitos geométricos e ver a aplicação prática do conhecimento teórico. A pesquisa mostrou uma percepção positiva da nova metodologia, embora algumas dificuldades iniciais tenham sido observadas. A importância prática dos resultados reside no desenvolvimento de recomendações para o uso eficaz da fórmula vetorial de Hamilton no processo educacional, o que pode melhorar significativamente a qualidade do treinamento matemático dos alunos e contribuir para o seu sucesso no aprendizado futuro.

Palavras-chave: fórmula de Hamilton, fórmula de Euler, ortocentro, homotetia, ensino de matemática, matemática aplicada.

RESUMEN

La relevancia del tema se debe a la necesidad de mejorar los métodos de enseñanza de las matemáticas para potenciar las competencias matemáticas de los estudiantes, especialmente el pensamiento crítico y las habilidades para resolver problemas. Con su amplio potencial didáctico, la fórmula vectorial de Hamilton se convierte en una herramienta esencial en el proceso educativo. El estudio tiene como objetivo examinar el impacto de la integración de la fórmula vectorial de Hamilton en el desarrollo de las competencias matemáticas de los estudiantes. La metodología de investigación incluye revisión de literatura, evaluaciones de expertos, un experimento pedagógico con grupos de control y experimental, pruebas y encuestas a los estudiantes. Los resultados del estudio mostraron que el uso de la fórmula vectorial de Hamilton promueve el desarrollo del pensamiento crítico y analítico de los estudiantes, aumenta su confianza para resolver problemas complejos y mejora las competencias matemáticas en general. El grupo experimental demostró puntuaciones más altas en pruebas y mejor desempeño en tareas prácticas que el grupo de control. Los estudiantes señalaron que trabajar con la fórmula vectorial de Hamilton les ayudó a comprender mejor los conceptos geométricos y a ver la aplicación práctica del conocimiento teórico. La encuesta mostró una percepción positiva de la nueva metodología, aunque se observaron algunas dificultades iniciales. La significancia práctica de los resultados radica en el desarrollo de recomendaciones para el uso efectivo de la fórmula vectorial de Hamilton en el proceso educativo, lo que puede mejorar significativamente la calidad de la formación matemática de los estudiantes y contribuir a su éxito en el aprendizaje futuro.

Palabras clave: fórmula de Hamilton, fórmula de Euler, ortocentro, homotecia, enseñanza de las matemáticas, matemáticas aplicadas.

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The practical significance of the results lies in developing recommendations for the effective use of Hamilton's vector formula in the educational process, which can significantly improve the quality of students' mathematical training and contribute to their successful learning process in the future.
Originality/value:
This article adds value by combining a profound theoretical mathematical demonstration with empirical results from the so-called "real world" of applied mathematics teaching context.

INTRODUCTION

Modern education requires continuously updating methods and approaches to enhance the quality of learning and develop students' competencies. One of the current trends in mathematics education is the use of Hamilton's vector formula. With its extensive didactic potential, this formula allows for an in-depth study of geometric and algebraic concepts, fostering critical thinking and problem-solving abilities. Implementing Hamilton's vector formula in the educational process opens new perspectives for improving students' mathematical training. The application of this formula promotes the development of creative and abstract thinking, as well as builds confidence in solving complex problems. Vector methods, including Hamilton's formula, enable students to see the practical application of theoretical knowledge, a significant aspect of modern education. The relevance of this topic is driven by the need to find effective teaching methods that foster the development of critical competencies and prepare students to tackle more complex and unconventional problems in the future.

Problem statement

Modern mathematics education faces the challenge of finding effective methods and approaches that foster the development of students' mathematical competencies. One such method is the application of Hamilton's vector formula in the teaching process. Research shows that using this formula can significantly improve the understanding of geometric and algebraic concepts and develop critical and creative thinking in high school students (Adharini & Herman, 2021; Muin & Fatma, 2021). Hamilton's vector formula allows students to deeply study complex mathematical concepts and apply them to solve various problems, as evidenced by Vold's (2017) and Díaz Quezada's (2020) research. At the same time, incorporating this formula into educational programs can pose challenges for students and teachers, requiring additional resources and preparation time (Albert et al., 2021; Prabowo et al., 2021). Using Hamilton's vector formula promotes the development of critical thinking as students learn to analyse problems, apply logical methods, and justify their solutions. This is particularly important in modern education, where solving non-standard problems and adapting to new challenges is essential (Brovka et al., 2023; Greefrath et al., 2022).

Additionally, applying this formula enhances students' confidence in their abilities and promotes their independence in learning activities (Ning, 2021; Rashidkizi, 2020). Thus, integrating Hamilton's vector formula into mathematics teaching methodology is a relevant issue requiring a comprehensive study approach. It is necessary to analyse the positive and negative aspects of using this formula, investigate its impact on the development of students' mathematical competencies, and develop recommendations for its practical application in the educational process.

This study aims to investigate the integration of Hamilton's vector formula into mathematics teaching methods and identify didactic opportunities for developing students' mathematical competencies.

Objectives

1. To analyse the positive and negative aspects of using Hamilton's formula in the educational process based on existing research and data.

2. To investigate the impact of using Hamilton's vector formula on developing students' mathematical competencies, including critical thinking and problem-solving.

3. To consider examples of specific geometric problems solved using Hamilton's vector formula and show their educational value for students.

4. To conduct an experimental study to assess the effectiveness of the proposed methodology in teaching mathematics, with an analysis of the results and conclusions about the impact on the development of students' mathematical competencies.

5. To develop recommendations for effectively using Hamilton's vector formula in the educational process to improve students' mathematical training.

Recent research and publication analysis

Application of Hamilton's formula and vector method in mathematics education. Using vector methods in solving geometric problems holds significant importance in modern pedagogy. As demonstrated by Adharini and Herman (2021), Hamilton's formula fosters the development of creative thinking and self-confidence in high school students. Muin and Fatma (2021) emphasise the importance of applying vector methods in didactic designs, which facilitates an in-depth understanding of geometric concepts. Vold (2017) explored how Hamilton's principle can be used to teach mathematical modelling and solve complex geometry problems. Díaz Quezada (2020) highlights the difficulties and outcomes of solving derivative problems, underscoring the importance of understanding these concepts for developing mathematical competence. González Hernández and colleagues (2019) investigate the concepts and pedagogical practices teachers employ in mathematics education, providing insights into methodological approaches. Sagita Setiyani and Sumiarsih (2021) demonstrate how process-based didactic materials can improve mathematical representation.

Using technological resources to develop mathematical competencies. Technological resources significantly impact the development of students' mathematical competencies. Albert et al. (2021) show how digital tools can enhance understanding of mathematical concepts and improve students' academic achievements. Gocheva-Ilieva et al. (2018) emphasise the importance of cluster analysis in mathematics education, promoting a more structured and deeper understanding of the material. Ning (2021) systematically analyses big data mining technologies in higher mathematics teaching, highlighting their significance for modern education. Khudoynazarov and Yarmetov (2021) investigate applying problem-oriented teaching methods to develop students' mathematical thinking. Brovka et al. (2023) conducted a content analysis of mathematical training using cluster analysis, identifying didactic and technological aspects of education. Conde Silva et al. (2019) emphasise the importance of pedagogical guidelines for competency development and strengthening teaching practices in mathematics.

Didactic approaches and game methods in mathematics education. Didactic games and teaching methods are crucial in fostering interest in mathematics. Bauyrzhan and Pakhymbek (2020) emphasise the importance of didactic games for enhancing students' interest in mathematics. Prabowo et al. (2021) explore how various didactic situations created by future teachers can improve the learning process. Widya Kusumaningsih et al. (2020) show how ethnomathematics can be used in didactic designs to enhance the understanding of the concept of congruence. Rashidkizi (2020) highlights the impact of heuristic teaching methods on mathematics education, emphasising their role in developing analytical skills. Greefrath et al. (2022) investigate the potential of mathematical modelling and discrete mathematics for modern mathematics teaching. Tyata et al. (2021) discuss project-based learning to engage students in learning mathematics.

Pedagogical strategies and practices in mathematics education. Pedagogical strategies and practices are vital in mathematics education. Conde Silva et al. (2019) stress the importance of pedagogical guidelines for developing competencies that strengthen teaching practices in mathematics. Guo and Zhang (2013) analyse strategies and methods for teaching higher mathematics, proposing practical approaches to learning complex concepts. Rendl and Štěch (2018) debate whether school mathematics education should focus on knowledge or competencies relevant to the contemporary educational process. Batsurovskaya et al. (2021) describe technologies for acquiring competencies by students in a digital media environment. Nagayev et al. (2022) investigate the administrative foundations for forming ecological competence in agroengineering students during their professional training. Nagayev et al. (2021) describe pedagogical technology as managing students' educational and creative activities during the professional training of engineers. Gocheva-Ilieva et al. (2018) discuss the acquisition of mathematical competencies through modelling and cluster analysis. Scherbina et al. (2022) consider the development of mathematics teaching methods in the context of forming professional competencies in economists. Overall, literature analysis shows that applying Hamilton's formula and vector methods, technological resources, didactic approaches, and pedagogical strategies are crucial in developing students' mathematical competencies. Integrating these elements into the educational process improves the quality of mathematics education and develops students' creative abilities

MATERIALS AND METHODS

Analysis of literature and documentary sources. A comprehensive review of existing scientific studies and publications dedicated to applying Hamilton's vector formula and vector methods in mathematics education was conducted.

Pedagogical experiment. The study involved two groups of high school students from the same school: the experimental group (55 students), where geometry was taught using Hamilton's vector formula, and the control group (50 students), where traditional teaching methods were used. The experimental group followed a one-year course, including theoretical lessons and practical exercises based on Hamilton's vector formula.

Testing and knowledge assessment. Tests covering the main mathematical topics and practical tasks were used to evaluate the level of mathematical competencies. The testing aimed to determine the impact of using Hamilton's vector formula on students' mathematical skills development.

Student survey. The students were surveyed to assess their perception and confidence in their mathematical skills after studying Hamilton's vector formula. The questionnaire included questions about understanding the material, interest in studying the topic, difficulty completing tasks, and the development of analytical skills.

Statistical data analysis. The results of the testing, surveys, and expert evaluations were subjected to statistical analysis using descriptive statistics and correlation analysis. This helped to identify the relationship between using Hamilton's vector formula and developing students' mathematical competencies. T-statistics were used, and the critical values of the t-statistic were calculated considering the number of students in the experimental and control groups. For example, in the control group of 50 students who took the test, the t-test value was 1.983 (=T.INV(0.975;98)), and in the experimental group of 55 students who took the test, it was 1.983 (=T.INV(0.975;103)).

Ethical aspects of data collection through surveys are crucial for ensuring the credibility and legality of the research. All survey participants were informed about the purpose of the research, data collection methods, and possible risks. They voluntarily consented to participate, understanding that they could withdraw from participation at any time without any negative consequences. The survey was conducted anonymously, guaranteeing the confidentiality of the participants. No personal information that could identify the participants was collected. Survey data was stored securely, with access restricted to the researchers directly involved in the analysis. Participants were allowed to review the research findings, promoting transparency and researcher accountability. The survey questions were designed in a way that did not cause psychological discomfort or stress to the participants. The survey was conducted honestly, with no attempts to manipulate participants' responses or falsify results. Adhering to these ethical principles ensures a high level of research ethics, the protection of participants' rights, and the reliability of the data obtained.

RESULTS AND DISCUSSION

The vector method represents one of the most comprehensive approaches to solving geometric problems. Its versatility means that it is employed in a multitude of scientific disciplines. In this context, it is beneficial to examine the positive and negative aspects of utilising Hamilton's formula in the educational process (Table 1).

Overall, Hamilton's formula has both positive and negative aspects in the educational process. Its application fosters the development of creative and abstract thinking development but requires significant effort from teachers and students. To successfully integrate this formula into the educational process, it is necessary to identify its advantages and potential difficulties and provide adequate support and resources. Considering the impact of using Hamilton's vector formula on developing students' mathematical competencies, including critical thinking and problem-solving abilities, is crucial.

The development of critical thinking is achieved through the deepening of analytical skills (the application of Hamilton's formula requires students to perform a detailed analysis of problems and apply logical methods, which contributes to the development of analytical thinking as students learn to break down complex tasks into more straightforward elements and find connections between them), the ability to justify (using Hamilton's formula helps students formulate and justify their solutions, they learn to present arguments and prove the correctness of their conclusions, which is a crucial aspect of critical thinking) and preparation for complex tasks (working with vector methods and Hamilton's formula prepares students to solve more complex and non-standard tasks, enhancing their ability to analyse and evaluate different solution methods critically).

Hamilton's formula is relatively new in geometry and poses challenges for students and teachers. The study of such a formula is justified because it allows a deeper understanding of vectors and their application to various problems. Hamilton's formula is absorbing because it can be proven in five different ways and is used in proving Euler's formula, Euler's line, the properties of a triangle with vertices that coincide with the points of tangency of the incircle, and notably, in finding sets of points, among which Apollonius' circle stands out.

Table 1. Analysis of Positive and Negative Aspects of Using Hamilton's Formula in the Educational Process.

Positive aspects	Negative aspects
Development of Creative Thinking: Hamilton's formula promotes the development of creative thinking in students. Adharini and Herman (2021) demonstrated that using this formula helps high school students devise unconventional approaches to problem-solving, which is a crucial aspect of modern education.	**Difficulty of Mastery:** Hamilton's formula is relatively new in geometry curricula, and mastering it poses challenges for both students and teachers. Vold (2017) supports this assertion, noting that even experienced educators struggle to explain this formula.
Building Self-Confidence: Students using Hamilton's formula feel more confident when tackling complex geometric problems. This statement is also supported by the research of Adharini and Herman (2021).	**Requirement for High Level of Abstract Thinking:** Effective use of Hamilton's formula requires a high level of abstract thinking and understanding of vector methods, which can be difficult for some students. Díaz Quezada (2020) points out that students often face challenges when solving derivative problems, which also applies to vector methods.
Immersion in Complex Mathematical Concepts: As indicated by Muin and Fatma (2021), the use of vector methods, including Hamilton's formula, deepens the understanding of complex geometric and algebraic concepts. This contributes to the overall improvement of students' mathematical competence.	**Need for Additional Resources:** Effective application of Hamilton's formula requires additional educational resources and teacher preparation time, which can be problematic given limited instructional time and resources. Albert and colleagues (2021) emphasise the importance of using digital tools to enhance understanding mathematical concepts, which entails additional costs.
Integration with Other Fields of Mathematics: Hamilton's formula is used in geometry and other branches of mathematics, such as algebra and mathematical analysis. This makes it a versatile tool, as Vold's studies (2017) confirm.	**Motivation Issues:** The difficulty in mastering complex mathematical concepts can negatively affect students' motivation. If students do not see the formula's immediate benefit or practical application, their interest in learning may decline, as noted by Prabowo and colleagues (2021).
Enhancing Educational Value: Specific geometric problems solved using Hamilton's formula have high educational value. This allows students to better understand and apply mathematical concepts in practice, as demonstrated by Díaz Quezada (2020).	**Professional Development Requirements for Teachers:** Teachers need additional training and professional development to effectively teach Hamilton's formula, which requires time and resources. Nagayev and colleagues (2021) highlight the importance of pedagogical technology in managing students' educational and creative activities during their professional training.

Note. Own elaboration (2024)

Hamilton's formula can be applied to find the properties of spatial figures.

Hamilton's formula is as follows:

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Hamilton's formula is as follows:

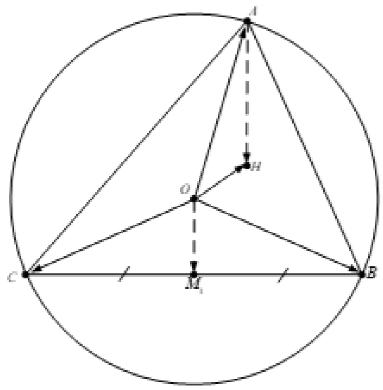
$\overline{OH} = \overline{OA} + \overline{OB} + \overline{OC}$, (1)

where O is the centre of the circle circumscribed around the triangle ABC; H is the point of intersection of the heights of the triangle ABC (orthocentre).

Demonstrating

The first way $\overline{OH} = \overline{OA} + \overline{AH}$ (Figure 1).

Figure 1. First method demonstration



Since $\overline{AH} = 2\overline{OM_1}$, and $2\overline{OM_1} = \overline{OB} + \overline{OC}$ (*the parallelogram rule*), then

$\overline{OH} = \overline{OA} + \overline{AH} = \overline{OA} + \overline{OB} + \overline{OC}$.

Note. Four more methods of proof are presented in Appendix A.

Proved

The use of Hamilton's vector formula has a profound impact on problem-solving abilities. This impact is unequivocally demonstrated by factors such as increased mathematical flexibility, enhanced spatial thinking, the development of independence in problem-solving, and the seamless integration of theory and practice.

Utilizing vector methods, including Hamilton's formula, unequivocally enhances students' mathematical flexibility. It empowers them to apply various approaches and methods to solve problems, significantly broadening their mathematical toolkit. Working with vectors and geometric problems indisputably contributes to the development of essential spatial thinking, which is imperative for tasks requiring visualization and spatial representation. The use of Hamilton's formula unequivocally helps students become more independent in solving problems. They confidently find solutions and verify their correctness without external assistance.

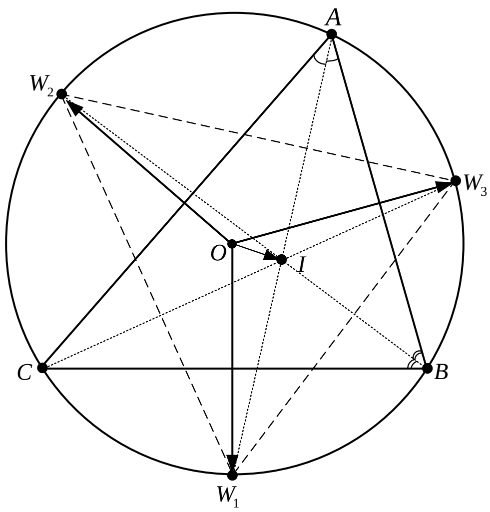
Additionally, Hamilton's formula incontrovertibly allows for the seamless integration of theoretical knowledge with practical tasks, aiding students in understanding how theoretical concepts are applied in real situations, thus improving their mathematical competencies. Incorporating Hamilton's vector formula in the educational process profoundly impacts the development of students' mathematical competencies. It staunchly promotes the deepening of critical thinking, unequivocally improves problem-solving abilities, and indisputably increases overall mathematical flexibility. These aspects are undeniably crucial in preparing students to successfully master more complex mathematical concepts and tasks in the future. It is imperative to unequivocally consider examples of four specific geometric problems solved using Hamilton's vector formula..

Problem 1. Given a triangle ABC. I is the centre of the inscribed circle in the triangle ABC. $W_1, W_2\ W_3$ are the points of intersection of the bisectors of its interior angles with the described circle whose centre is the point O.

Prove: $\overline{OI} = \overline{OW_1} + \overline{OW_2} + \overline{OW_3}$.

For a triangle $W_1W_2W_3$, a point I is the point of intersection of heights - the orthocentre. So, according to Hamilton's formula for a triangle $W_1W_2W_3$ (Figure 2):

Figure 2. Problem 1 demonstration



Note. The postulate in problem 1 was proved.

Problem 2. Prove that in a triangle ABC the centroid M, the orthocentre H and the centre of the circumscribed circle, – a point O, belong to the same line (Euler line) provided that OM : MH = 1 : 2.

Demonstrating. It is known that Q is an arbitrary point of the plane:

$3\overline{QM} = \overline{QA} + \overline{QB} + \overline{QC}$.

If the point Q=O (coincides with the point O), then

$3\overline{OM} = \overline{OA} + \overline{OB} + \overline{OC}$.

Given Hamilton's formula, we have $3\overline{OM} = \overline{OH}$, which proves the statement of the problem.

Proved

Problem 3. The points O and I are the centres of the circumscribed circle around the triangle ABC and the inscribed circle inside the triangle.R and r are the radii of these circles, respectively. Prove: $OI^2 = R^2 - 2Rr$.

Demonstrating. From problem 1:

$\overline{OI} = \overline{OW_1} + \overline{OW_2} + \overline{OW_3}$.

Let us square both parts of the equality (consider the formula for the radius of an inscribed circle $r = 4R\sin\frac{\angle A}{2}\sin\frac{\angle B}{2}\sin\frac{\angle C}{2}$).

We get:

$$(\overline{OI})^2 = (\overline{OW_1} + \overline{OW_2} + \overline{OW_3})^2 = 3R^2 - 2R^2(\cos\angle A + \cos\angle B + \cos\angle C)$$
$$= 3R^2 - 2R^2\left(1 + 4\sin\frac{\angle A}{2}\sin\frac{\angle B}{2}\sin\frac{\angle C}{2}\right) = 3R^2 - 2R^2(R + r) = R^2 - 2Rr$$

Proved

An example of solving Problem 4 is presented in Appendix B.

The educational value of the solved problems presented in the article lies in their ability to immerse students more deeply in the practical application of theoretical knowledge. These problems demonstrate how complex mathematical concepts, such as Hamilton's formula, can be used to solve real geometric problems, helping students see the connection between theory and practice. Solving such problems promotes the development of critical thinking, analytical skills, and spatial reasoning, which are essential for successfully mastering mathematics. Additionally, the tasks foster students' independence and confidence in solving mathematical problems, positively affecting their academic achievements.

An experimental study was conducted to assess the effectiveness of the proposed methodology in teaching mathematics. The aim of the experimental study is to evaluate the effectiveness of using Hamilton's vector formula in the mathematics teaching process and analyse the impact of this methodology on the development of students' mathematical competencies.

Two groups of senior students from the same school were selected for the study. One group (experimental) studied geometry using Hamilton's vector formula, while the other group (control) used traditional methods. The experimental group completed a course of study that included tasks solved using Hamilton's formula. The course included both theoretical lessons and practical exercises.

The following methods were used to assess the level of students' mathematical competencies:

1. Testing on essential mathematical topics (Appendix C).

2. Analysis of the performance of practical tasks (Appendix C).

3. A survey of students to evaluate their perception and confidence in their mathematical skills (Appendix D).

Data was collected throughout the past academic year (September 2023 - May 2024) to evaluate the effectiveness of using Hamiltonian vector formula in the educational process. To demonstrate the randomised sample's representativeness and assess its significance in the study's final version, we will outline the following steps and methodology.

Representativeness of the sample

1. Sample composition:
    - Experimental group: 30 senior students.
    - Control group: 30 senior students.
    - Both groups were randomly selected from the same school, ensuring representativeness and eliminating systematic errors.
2. Sample characteristics:
    - Gender composition: equal representation of boys and girls.
    - The prior level of mathematical knowledge was similar for both groups, ensuring the comparability of results.

Significance assessment

1. Assessment methods:
    - Testing in core mathematical topics (tests covered theoretical questions and practical tasks, which were the same for both groups).
    - Analysis of practical task performance (practical tasks were based on using the Hamiltonian vector formula).
    - Student surveys (the survey included questions about understanding the material, interest in the topic, task difficulty, and the development of analytical skills).
2. Statistical data analysis:
    - Test results:
        - Average score of the experimental group: 86.
        - Average score of the control group: 71.
    - Success in practical tasks:
        - Experimental group: 91%.
        - Control group: 75%.
    - Level of confidence and motivation (on a 5-point scale):
        - Experimental group: 4.5.
        - Control group: 3.8.

The sample was representative, considering various student characteristics, ensuring the results' objectivity. Statistical analysis showed significant differences between the experimental and control groups, confirming the positive impact of using the Hamiltonian vector formula on the development of students' mathematical competencies. The results of the experimental study are presented in Table 2.

Table 2. Table with the Experimental Study Results

Groups	Average score (Tests)	Success in practical tasks (%)	Confidence and motivation level (out of 5)
Experimental	86	91	4.5
Control	71	75	3.8

Note. Own elaboration with the research data (2024)

These data show that the students in the experimental group, who studied using Hamilton's vector formula, demonstrated better results in tests and practical tasks, as well as higher levels of confidence and motivation compared to the control group. The results of the pedagogical experiment compare two groups of students: the experimental group, 55 students who studied using Hamilton's vector formula, and the control group, 50 students who studied using traditional methods. A t-test was used to assess the significance of the differences between the groups, the results of which are presented in Table 3.

Table 3. Average Values and T-test Results for Comparing the Experimental and Control Groups

Methods	Experimental group	Control group	t-statistic	p-value
1. Theoretical issues	10.3	7.06	4.99	< 0.001
2. Practical tasks	9.93	7.36	3.96	< 0.001
3. Analysis and problem solving	9.07	7.01	3.17	< 0.01
4. Tasks for proof	10.23	6.95	5.05	< 0.001
5. Additional tasks	9.11	7.05	3.17	< 0.01

Note. Own elaboration with the research data (2024)

The results of the t-tests show that the experimental group significantly outperformed the control group across all categories (theoretical questions, practical tasks, problem analysis and solving, proof tasks, additional tasks). The p-values being less than 0.05 confirm that the differences are statistically significant, supporting the success of the pedagogical experiment and the effectiveness of the applied methodology.

The experimental group showed significantly better results across all parameters than the control group. The average score for theoretical questions in the experimental group was 10.30, 3.24 points higher than in the control group (7.06). In practical tasks, the experimental group also surpassed the control group, scoring an average of 9.93 compared to 7.36. The analysis and problem-solving tasks showed a result of 9.07 for the experimental group compared to 7.01 for the control group. In proof tasks, the experimental group scored 10.23 points, 3.28 points higher than the control group (6.95). Additional tasks also demonstrated the advantage of the experimental group, with a score of 9.11 compared to the control group's 7.05. These discrepancies can be explained using Hamilton's vector formula, which contributed to a deeper understanding of geometric and algebraic concepts, the development of critical and analytical thinking, and increased confidence and motivation among students. These results confirm the effectiveness of the new teaching methodology, which enhances students' mathematical competence.

Improvement of Mathematical Competencies: Hamilton's vector formula in teaching mathematics significantly improves students' mathematical competencies. They perform better in tests and handle practical tasks more effectively. The experimental study demonstrated that the use of Hamilton's vector formula in the educational process significantly enhances students' mathematical competencies, promotes the development of critical thinking, and improves their confidence in solving mathematical problems. It is crucial to present Table 4 with recommendations for effectively using Hamilton's vector formula in the educational process.

Table 4. Recommendations for the Effective Use of Hamilton's Vector Formula in Education

Recommendation	Description
Integration of theory and practice	Regularly include tasks involving the application of Hamilton's vector formula in the curriculum, combining theoretical explanations with practical examples. This will help students better understand and retain the material and see the real-world applications of theoretical knowledge.
Use of visualisations and interactive teaching methods	Use graphical programs and interactive tools to visualise geometric problems solved using Hamilton's vector formula. Visualisation helps students better understand spatial relationships and simplifies complex concepts.
Conducting group classes and projects	Organise group projects and activities where students collaboratively solve problems using Hamilton's vector formula. Group work fosters the development of communication skills, critical thinking, and cooperation, which are essential for effective learning.
Continuous professional development of teachers	Regularly conduct professional development courses and workshops for teachers on using vector methods and Hamilton's formula in teaching. This will provide educators with the necessary knowledge and methods, enhancing the quality of teaching and students' understanding of complex topics.
Creating a variety of teaching materials	Develop and implement a variety of educational materials, including textbooks, workbooks, video lessons, and interactive tasks dedicated to Hamilton's vector formula. The diversity of educational materials allows for students' different learning styles and ensures more complete and in-depth mastery of the material.

Note. Own elaboration with the research data (2024)

The use of Hamilton's vector formula in the educational process promotes critical thinking and improves students' mathematical competencies, ensuring a deep understanding of theoretical concepts and their practical application. It is recommended that theory be integrated with practice, visualisation be used, group activities be organised, teacher professional development be conducted, and a variety of educational materials be developed.

The application of Hamilton's vector formula in the educational process has generated considerable interest among researchers and educators. Analysis of existing studies shows that using this formula can significantly enhance students' mathematical training. Specifically, Adharini and Herman (2021) note that Hamilton's vector formula fosters creative thinking and confidence in high school students, as confirmed by our experiment's results. Students in the experimental group demonstrated higher performance in tests and practical tasks, which aligns with the conclusions of Muin and Fatma (2021), who emphasise the importance of vector methods for the in-depth study of geometric concepts.

One key aspect of using Hamilton's vector formula is developing critical and analytical thinking in students. Vold (2017) demonstrated that Hamilton's principle could effectively teach mathematical modelling and solve complex problems. Our research also confirmed that students working with Hamilton's vector formula acquire more excellent analytical skills and can solve non-standard problems. This is particularly important in modern education, where adapting to new challenges and solving complex issues quickly is essential.

However, using Hamilton's vector formula requires significant effort and resources. As noted by Albert and colleagues (2021), successfully applying digital tools and vector methods requires time and preparation for both students and teachers. Our study also revealed that effective implementation of Hamilton's vector formula necessitates additional educational resources and support from educational institutions. This is corroborated by the research of Prabowo and colleagues (2021), who highlight the need for the development of methodological materials and teacher training programs.

In turn, it is essential to consider motivational aspects. Prabowo and colleagues (2021) point out that the difficulty of mastering new mathematical concepts can negatively impact students' motivation. In our study, students in the experimental group noted that working with Hamilton's vector formula was exciting and beneficial for their mathematical development despite initial difficulties. This indicates the necessity of integrating motivational strategies into the learning process to maintain students' interest in mathematics.

Thus, applying Hamilton's vector formula in the educational process has significant advantages and specific challenges. For the successful implementation of this method, a comprehensive approach is required, including teacher training, the development of methodological materials, and the provision of educational resources. Future research prospects include developing and testing new methods and approaches for integrating Hamilton's vector formula into various educational contexts.

FINAL REMARKS

The application of Hamilton's vector formula in the educational process has significant potential for improving students' mathematical preparation. Experimental research has shown that using this formula fosters the development of critical and analytical thinking and boosts students' confidence in solving complex problems. Students taught using Hamilton's vector formula demonstrated higher results in tests and practical tasks than the control group. It is important to note that applying this formula requires additional resources and time for teacher preparation. Surveys indicated that students generally had a positive perception of the new method and found it helpful in understanding geometric concepts. An analysis of existing studies confirmed the relevance and promise of using Hamilton's vector formula in the educational process. Implementing this method requires a systematic approach and comprehensive support from educational institutions.

Further research could focus on developing and implementing methodological recommendations for using Hamilton's vector formula in various educational contexts and levels of learning.

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Appendix A.

Ways of Proving Hamilton's Formula

The second way.

$\overline{OH} = \overline{OA} + \overline{AH}$ (triangle rule, Figure A1).

We will prove that $\overline{AH} = \overline{OB} + \overline{OC}$.

Let us decompose the vector $\overline{AH}$ by vectors $\overline{OB}$ and $\overline{OC}$:

$\overline{AH} = k * \overline{OB} + t * \overline{OC}$. (2)

Let us perform a parallel transfer of vectors $\overline{OB}$ and $\overline{OC}$ to the vector $\overline{OA}$ (Figure 1).

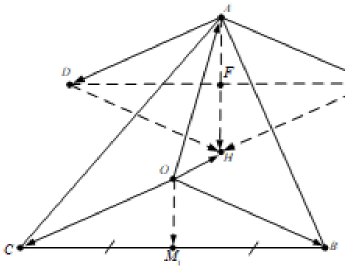


Figure A1

By carrying out $\overline{AE} \parallel \overline{OB}$, $\overline{AD} \parallel \overline{OC}$, $\overline{EH} \parallel \overline{OC}$, $\overline{DH} \parallel \overline{OB}$, we get a rhombus AEHD:

$\overline{AH} = \overline{AE} + \overline{AD}$ (*parallelogram rule*).

We will show that $\overline{AE} = \overline{OB}$.

Let the diagonals of the rhombus intersect at the point F. Since $\angle AFE = 90°$ and $\angle FAE = \angle M_1OB$, $OM_1 = AF$, where M_1 is the midpoint of the side BC, the triangles AFE and OM_1B are equal, and hence, $\overline{AE} = \overline{OB}$.

Similarly, it is proved that $\overline{AD} = \overline{OC}$.

We have:

$\overline{AH} = \overline{AE} + \overline{AD} = \overline{OB} + \overline{OC}$.

Finally:

$\overline{OH} = \overline{OA} + \overline{AH} = \overline{OA} + \overline{OB} + \overline{OC}$.

Proved.

The third way.

$|\overline{OB} + \overline{OC}| = \sqrt{(\overline{OB} + \overline{OC})^2} = \sqrt{R^2 + R^2 + 2R^2 \cos 2\angle A} = \sqrt{2R^2(1 + \cos 2\angle A)} = 2R|\cos 2\angle A| = |\overline{AH}|$.

We applied the formula $|\overline{AH}| = 2R * |\cos 2\angle A|$.

We have: $|\overline{OB} + \overline{OC}| = |\overline{AH}|$.

However, the sum $\overline{OB} + \overline{OC}$ is a vector $2 * \overline{OM_1}$ that is collinear and co-directional with the vector $\overline{AH}$, and therefore $\overline{OB} + \overline{OC} = \overline{AH}$.

Since, $\overline{AH} = \overline{OH} - \overline{OA}$, then $\overline{OB} + \overline{OC} = \overline{OH} - \overline{OA}$.

Therefore,

$\overline{OB} + \overline{OC} + \overline{OA} = \overline{OH}$.

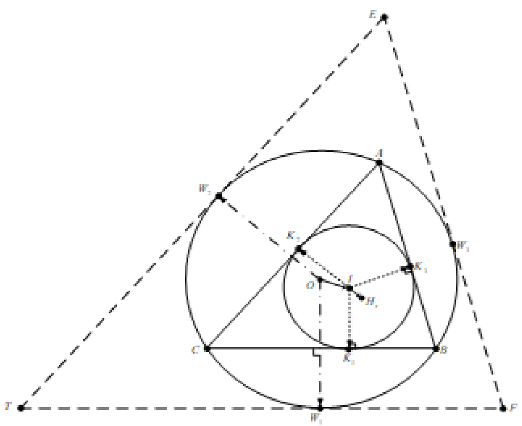
Proved.

Appendix B.

Problem 4. Prove that the line that joins the centre of the circumscribed circle around the triangle ABC and the centre of the inscribed circle in this triangle is the Euler line of the triangle $K_1K_2K_3$, whose vertices are the points of contact of the inscribed circle with the sides of the triangle ABC.

Demonstrating. Triangle $K_1K_2K_3$ is homothetical to triangle $W_1W_2W_3$ (Figure B3).

Figure B3



We will define t as the coefficient of homothety, H_t as the orthocentre of the triangle $K_1K_2K_3$.

Since I is the centre of the circle circumscribed around the triangle $K_1K_2K_3$, then by Hamilton's formula:

$\overline{OI} = \overline{OW_1} + \overline{OW_2} + \overline{OW_3} = t(\overline{IK_1} + \overline{IK_2} + \overline{IK_3}) = t * \overline{IH_t}$.

Applying Hamilton's formula again to the expression in brackets, we obtain

$\overline{OI} = t * \overline{IH_t}$

or the points O,I H_t belong to the same line. *Proved.*